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DIALOGUES AS A FOUNDATION FOR INTUITIONISTIC LOGIC

SUMMARY OF CONTENTS

The principal content of this article is a (new) foundation for intuitionistic logic, based on an analysis of argumentative processes as codified in the concepts of a *dialogue* and a *strategy* for dialogues. This work is presented in Section 3. A general historical introduction is given in Section2. Since already there the reader will need to know exactly what a dialogue and a strategy shall be, these basic concepts are defined in the (purely technical) Section 1.

1 BASIC CONCEPTS: DIALOGUES AND STRATEGIES

I consider a first-order language, built with variables $x, y, \ldots$ and terms t; formulas shall be constructed from atomic formulas with the propositional connectives $\wedge, \vee, \rightarrow, \neg$ and the quantifiers $\forall, \exists$; I shall also consider $\vee, \wedge_1, \wedge_2, \exists$ as *special symbols* in their own right. By an *expression* I understand either a term or a formula or a special symbol. I introduce two further symbols P and Q; taking two new (and disjoint) copies of the set of expressions, I form for every expression e two new expressions Pe and Qe, the *P-signed* and the *Q-signed version* of the expression e.

The symbols P, Q shall symbolise two persons engaged in an argument or in a dialogue; I shall use X, Y as variables for P, Q and shall assume $X \neq Y$. An *argumentation form* is a schematic presentation of an argument, concerning a logically composite assertion; it describes how a composite assertion made by C may be *attacked* by Y and how, if possible, this attack may be *answered* by X. As the logical form of the composite assertion shall completely determine the argument, each of the four propositional connectives and each of the two quantifiers determines an argumentation form:

D.M. Gabbay and F. Guenthner (eds.),
Handbook of Philosophical Logic, 2nd Edition, Volume 5, 115–145.

$\wedge$:	assertion:	$Xw_1 \wedge w_2$	
	attack:	$Y\wedge_i$	(i.e., Y chooses $i = 1$ or $i = 2$)
	answer:	Xw_i	
$\vee$:	assertion:	$Xw_1 \vee w_2$	
	attack:	$Y\vee$	
	answer:	Xw_i	(i.e., X chooses $i = 1$ or $i = 2$)
$\rightarrow$:	assertion:	$Xw_1 \rightarrow w_2$	
	attack:	Yw_1	
	answer:	Xw_2	
$\neg$:	assertion:	$x\neg w$	
	attack:	Yw	
	answer:	*no answer possible*	
$\forall$:	assertion:	$X\forall xw$	
	attack:	Yt	(i.e., Y chooses the term t)
	answer:	$Xw(t)$	
$\exists$:	assertion:	$X\exists xw$	
	attack:	$Y\exists$	
	answer:	$Xw(t)$	(i.e., X chooses the term t).

In the last two answers I have written $w(t)$ for the substitution instance obtained from w if the term t is substituted for the variable x.

A dialogue shall be a (finite or infinite) sequence δ of statements, i.e., signed expressions, stated alternatingly by P and Q and progressing in accordance with the argumentation forms; I shall consider only such dialogues which are begun by P. Since it is necessary to distinguish carefully between attacks, answers and the assertions they refer to, I shall introduce besides δ an accompanying sequence η of references, and there I shall use the symbols A for *attack* and D for *answer* (*defense*). For notational convenience, I shall assume that a natural number *is* the set of all smaller natural numbers (whence 0 is the first natural number), and a *sequence* shall always be a function, defined on either a natural number or on the set ω of all natural numbers. The precise definition then reads as follows:

A *dialogue* δ, η consists of two sequences such that

> δ is a sequence of signed expressions,
>
> η is a function defined on the *positive* members of def(δ), and if n in def(η) is an ordered pair $[m, Z]$ such that m is a natural number less than n and Z is either A or D ,

satisfying the properties $(D00)$-$(D02)$:

($D00$) $\delta(n)$ is P-signed if n is even and Q-signed if n is odd; $\delta(0)$ is a composite formula.

($D01$) If $\eta(n) = [m, A]$ then $\delta(m)$ is a composite formula and $\delta(n)$ is attack upon $\delta(m)$ according to the appropriate argumentation form.

($D02$) If $\eta(p) = [n, D]$ then $\eta(n) = [m, A]$ and $\delta(p)$ is the answer to the attack $\delta(n)$ according to the appropriate argumentation form.

The signed formulas occurring as values of δ are called the *assertions* of the dialogue while the remaining values of δ are *symbolic statements* or, more correctly, *symbolic attacks*. The numbers in def(δ) are called the *positions* or *places* of the dialogue. If Pv is the assertion $\delta(0)$, the dialogue is said to be a dialogue *for* the formula v (or, sometimes, for Pv).

Assume now that a particular class H of dialogues is given, defined maybe by additional conditions, which has the property that, for every position n of an H-dialogue δ, η, the *restrictions* of δ, η to positions i such that $i \leq n$ form an H-dialogue again. Assume further hat a subclass of H has been defined, consisting of certain *finite* H-dialogues which then are said to be the H-dialogues *won* by P. Let v be a composite formula; to say that P has an H-strategy shall mean that P is in possession of a system of information, consisting of possible choices of P-statements in dialogues, such that every H-dialogue for v is won by P if only P chooses, after every statement made by Q, its own statement from this system of information. In order to formulate a more precise definition, recall that a *tree* S is a partially ordered set of elements called *nodes* with the following properties: there exists a *largest* element e_S (the top node), and for every node e the number $\|e\|$ of nodes f such that $e \leq f < e_S$ is *finite*; every node except e_S has exactly one *upper neighbour* but may have arbitrarily many *lower neighbours* (i.e., the tree is branching downwards). A *path* in S is a linearly ordered subset of nodes which, together with each of its elements e, contains all the preceding nodes f with $e \leq f$; a *branch* is a path which is maximal. If A is a branch of S, let α_A be the unique order-preserving bijection which maps either a natural number or all of ω onto A, i.e. $\|\alpha_A(i)\| = i$ holds for every node $\alpha_A(i)$ in A. Consider now a tree S and functions δ, η where δ is defined on all nodes of S and η on the nodes different from e_S; for every branch A define $\delta_A = \delta \cdot \alpha_A$, $\eta_A = \eta \cdot \alpha_A$. The triplet S, δ, η then is an *H-strategy* for v if

($S0$) For every branch A of S the pair δ_A, η_A is an H-dialogue for v which is won by P.

($S1$) For every node e of S the following is the case. If $\|e\|$ is odd then S does not branch at e. If $\|e\|$ is even then e has as many lower neighbours as Q has possibilities to extend, by adding a new position, to an H-dialogue the (restricted) dialogue leading to e, and δ, η assign these lower neighbours the values which realise these possibilities.

The general definitions having been established, particular classes of dialogues can be introduced. To do so, I shall need the following terminology. Let δ, η be a dialogue, and let $\delta(n)$ be one of its attacks. The attack $\delta(n)$ will be said to be *open at a position* k with $n<k$ if there is no position n' with $n<n' \leq k$ which carries an answer $\delta(n')$ to that attack. In particular, an attack upon a formula $X\neg v$ remains open at all later places. A *D-dialogue* shall be a dialogue δ, η satisfying the following properties $(D10)$–$(D13)$:

$(D10)$ P may assert an atomic formula only after it has been asserted by Q before: if $\delta(n) = Pa$ and a is atomic then there exists m such that $m<n$ and $\delta(m) = Qa$.

$(D11)$ If, at a position $p-1$, there are several open attacks suitable to be answered at p, then only the *latest* of them may be answered at p : if $\eta(p) = [n, D]$ and if $n<n'<p, n'-n \equiv 0 \pmod 2$, $\eta(n') = [m', A]$ then there exists p' such that $n'<p'<p, \eta(p') = [n', D]$.

$(D12)$ An attack may be answered at most once: for every n there exists at most one p such that $\eta(p) = [n, D]$.

$(D13)$ A P-formula may be attacked at most once: if m is even then there exists at most one n such that $\eta(n) = [m, A]$.

A D-dialogue is said to be *won by* P if it is finite, ends with an even position and if the rules do not permit Q to continue with another attack or answer. In that case the last position carries an atomic formula asserted by P.

The importance of D-dialogues rests in the fact that the formulas for which there exist D-strategies are precisely those provable in intuitionistic logic. This follows from the following, stronger

EQUIVALENCE THEOREM. *There exist recursive algorithms which, for every formula* v*, transform a proof of the sequent* $\Rightarrow v$ *in Gentzen's calculus* LJ *(for intuitionistic logic) into a* D*-strategy — and vice versa.*

Contrary to first appearances, a proof of this theorem is by no mean obvious; it cannot be pursued here and may be found in Felscher [1981; 1985].

An *E-dialogue* shall be a D-dialogue satisfying the additional condition that Q can react only upon the immediately preceding utterance of P:

(E) For every n in def(δ): if n is odd then $\delta(n)$ is either attack upon $\delta(n-1)$ or answer to $\delta(n-1)$.

An E-dialogue is said to be *won by* P if, again, it is finite, ends with an even position and if now the rules for E-dialogues do not permit Q to continue with either an attack or an answer. There will be occasion to refer to the following result which is auxiliary to the proof of the Equivalence Theorem.

EXTENSION LEMMA. *There is a recursive algorithm by which every E-strategy can be embedded into a D-strategy.*

It follows from this lemma that the Equivalence theorem holds also for E-strategies in place of D-strategies.

Readers not familiar with the use of dialogues may appreciate the following *examples* in which $a, b, \ldots$ are assumed to be atomic formulas.

(1a)

0.	$P(a \wedge b) \to (a \wedge b)$	
1.	$Q(a \wedge b)$	[0,A]
2.	$P\wedge_1$	[1,A]
3.	Qa	[2,D]
4.	$P\wedge_2$	[1,A]
5.	Qb	[4,D]
6.	$P(a \wedge b)$	[1,D]

7.	$Q\wedge_1$	[6,Q]	7.	$Q\wedge_2$	[6,Q]
8.	Pa	[7,D]	8.	Pb	[7,D]

(1b)

0.	$P(a \wedge b) \to (a \wedge b)$	
1.	$Q(a \wedge b)$	[0,A]
2.	$P(a \wedge b)$	[1,D]

3.	$Q\wedge_1$	[2,A]	3.	$Q\wedge_2$	[2,A]
4.	$P\wedge_1$	[1,A]	4.	$P\wedge_2$	[1,A]
5.	Qa	[4,D]	5.	Qb	[4,D]
6.	Pa	[3,D]	6.	Pb	[3,D]

Here we have two different D-strategies for the same formula.

(2a)

0.	$P(a \to \neg\neg a)$	
1.	Qa	[0,A]
2.	$P\neg\neg a$	[1,D]
3.	$Q\neg a$	[2,A]
4.	Pa	[3,A]

(2b)

0.	$P(\neg\neg a \to a)$	
1.	$Q\neg\neg a$	[0,A]
2.	$P\neg a$	[1,A]
3.	Qa	[3,A]